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Higher Dimensional Unified Description of Early Universe with Variable Gravitational and Cosmological Constants

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Abstract In this paper we consider the five dimensional cosmological model with variable gravitational and cosmological constants in the presence of perfect fluid in Kaluza-Klein theory of gravitation. The exact solutions of the field equations are obtained by using the gamma law equation of state $p = (\gamma - 1)\rho$ in which the parameter γ depends on scale factor R. A unified description of early universe is presented with the assumption $\Lambda = \beta H^2$ in which *inflationary phase* is followed by *radiation dominated phase*. The various physical aspects of the cosmological models are also discussed in the framework of higher dimensional space time.

Keywords Higher dimensional space time · Early universe · Gravitational and cosmological constants

1 Introduction

The possibility that the world have more than the four dimensions is due to Kaluza [1] and Klein [2], who used one extra dimension to unify gravity and electromagnetism in a theory which was essentially five dimensional general relativity. This idea has been worked by a large number of people, who have found models for various phenomenon in particle physics and cosmology in five or more dimension [3–6]. Overuin and Wesson [7] have presented an excellent review of Kaluza-Klein theory and higher-dimensional unified theories, in which the cosmological and astrophysical implications of extra-dimension have been discussed. Also, many authors have studied Kaluza-Klein cosmological models with different matters

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[8–12]. There is now extensive literature dealing with different aspect of higher dimensional cosmologies.

A great number of exact cosmological solutions of Einstein field equations with different equation of state and different symmetries, including or not a cosmological constant, has been found with 5D [13–15] and also with arbitrary number of dimension [16–19]. Sahdev [20], Emelyanov et al. [21] and Chatterjee and Bhui [22], have studied physics of the universe in higher-dimensional space-time.

In relativistic and observational cosmology, the evolution of the universe is described by Einstein's field equations together with perfect fluid and an equation of state. Einstein's theory of gravity contains two parameters-Newtonian gravitational "constant" G and cosmological "constant" Λ. Normally, these are considered as fundamental constants. The gravitational "constant" G plays the role of a coupling constant between geometry of space and matter content in Einstein field equations. In an evolving universe, it appears natural to look at this constant as a function of time. There have been many extensions of Einstein's theory of gravitation, with time-dependent G, in order to achieve a possible unification of gravitation and elementary particle physics. Canuto et al. [23] made numerous suggestions based on different arguments that G is indeed time dependent. The Λ-term arises naturally in general-relativistic quantum field theory where it is interpreted as the energy density of the vacuum (see [24]). It is widely believed that the value of Λ was large during the early stages of evolution and strongly influenced its expansion, whereas its present value is incredibly small. The possibility of Λ as a function of time has been considered by Canuto et al. [25].

It is still a challenging problem to know the exact physical situation at very early stages of the formation of our universe. At the very early stages of evolution of universe, it is generally assumed that during the phase transition (as the universe passes through it's critical temperature) the symmetry of the universe is broken spontaneously. Carvalho [26] has studied Roberson-Walker model in general relativity by using equation of state $p = (\gamma - 1)\rho$, where the adiabatic parameter γ varies with cosmic time. His work motivate one to consider further work in some alternative theories of gravitation.

In this paper we have consider the Kaluza-Klein type cosmological model with variable G and Λ in the presence of perfect fluid and study the evolution of the universe as it goes from inflationary phase to radiation dominated phase. The possibility of G is also studies in both the cases. Some physical aspects of the model are also discussed in the context of higher dimensional space time.

2 Model and Field Equations

Let us consider the Kaluza-Klein type metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1-kr^2)d\psi^2\right], \tag{1}$$

where $R(t)$ is the scale factor and $k = 0. -1$, or $+1$ is the curvature parameter for flat, open and closed universe, respectively. The universe is assumed to be filled with distribution of matter represented by energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}, \tag{2}$$

where ρ is the energy density of the cosmic matter and p is its pressure and u_μ is the five-velocity vector such that $u_\mu u^\mu = 1$.

The Einstein field equations with time-dependent cosmological and gravitational constants is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G(t) T_{\mu\nu} + \Lambda(t) g_{\mu\nu}, \tag{3}$$

where $R_{\mu\nu}$ is the Ricci tensor, $G(t)$ and $\Lambda(t)$ being the variable gravitational and cosmological constants.

The divergence of (3), taking into account the Bianchi identity, gives

$$(8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu})^{;\nu} = 0. \tag{4}$$

Equations (3) and (4) may be considered as the fundamental equations of gravity with G and Λ coupling parameters. Using comoving coordinates

$$u_{\mu} = (1, 0, 0, 0, 0), \tag{5}$$

in (2) and with line element (1), Einstein's field (3) yields

$$8\pi G(t)\rho = \frac{6\dot{R}^2}{R^2} + \frac{6k}{R^2} - \Lambda(t), \tag{6}$$

$$8\pi G(t) p = -\frac{3\ddot{R}}{R} - \frac{3\dot{R}^2}{R^2} - \frac{3k}{R^2} + \Lambda(t), \tag{7}$$

where dot denotes derivative with respective to t.

In uniform cosmology $G = G(t)$ and $\Lambda = \Lambda(t)$ so that the conservation (4) is given by

$$\dot{\rho} + 4(\rho + p)H = -\left(\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G}\right). \tag{8}$$

The usual energy momentum conservation relation $T^{\mu\nu}_{;\nu} = 0$ leads to

$$\dot{\rho} + 4(\rho + p)H = 0. \tag{9}$$

Therefore, (8) yields

$$\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{10}$$

The field equations (6)–(7) can also be written as

$$\frac{3\ddot{R}}{R} = -4\pi G(t)\left[2p + \rho - \frac{\Lambda(t)}{8\pi G(t)}\right], \tag{11}$$

$$\frac{6\dot{R}^2}{R^2} = 8\pi G(t)\left[\rho + \frac{\Lambda(t)}{8\pi G(t)}\right] - \frac{6k}{R^2}. \tag{12}$$

Equations (11) and (12) can be rewritten in terms of the Hubble parameter $H = \frac{\dot{R}}{R}$ to give, respectively

$$\dot{H} + H^2 = -\frac{4\pi}{3} G(t)(2p + \rho) + \frac{\Lambda(t)}{6}, \tag{13}$$

$$H^2 = \frac{4\pi}{3} G(t)\rho + \frac{\Lambda(t)}{6} - \frac{k}{R^2}. \tag{14}$$

In order to solve the field equations (9), (10), (13) and (14) we assume that the pressure p and density ρ through the gamma law of equation of state

$$p=(\gamma-1)\rho, \tag{15}$$

where γ is an adiabatic parameter varying continuously with cosmological time so that in the course of its evolution the universe goes through a transition from an inflationary phase to radiation-dominated phase. Carvalho [26] assumed the functional form of γ depends on scale factor as

$$\gamma(R)=\frac{4}{3}\frac{A(\frac{R}{R_*})^2+(\frac{a}{2})(\frac{R}{R_*})^a}{A(\frac{R}{R_*})^2+(\frac{R}{R_*})^a}, \tag{16}$$

where A is a constant and a is free parameter related to the power of cosmic time and lies $0\le a<1$. Here R_* is certain reference value such that if $R\ll R_*$, inflationary phase of the evolution of the universe is obtained and for $R\gg R_*$, we have a radiation-dominated phase.

Using (15) into (13) we obtain

$$\dot{H}+H^2=-\frac{4\pi}{3}G(t)[(2\gamma-1)\rho]+\frac{\Lambda(t)}{6}. \tag{17}$$

Eliminating ρ between (14) and (17), we get

$$\dot{H}+2\gamma H^2+(2\gamma-1)\frac{k}{R^2}=\frac{\gamma\Lambda(t)}{3}. \tag{18}$$

Now consider $k=0$ (flat universe), (18) reduces to

$$H'+\frac{2\gamma H}{R}=\frac{\gamma\Lambda}{3HR}, \tag{19}$$

where a prime denotes differentiation with respect to R.

3 The Field Equation and Its Solution

We obtain the solution of (19) by taking certain assumptions on Λ.

We assume that

$$\Lambda(t)=\beta H^2, \tag{20}$$

where β is dimensionless positive constants.

By using this values of $\Lambda(t)$ in (19), we obtain

$$H'+\left(2-\frac{\beta}{3}\right)\gamma\frac{H}{R}=0. \tag{21}$$

Substituting the value of γ from (16) into (21) and integrating, we get

$$H=\frac{C}{[A(\frac{R}{R_*})^2+(\frac{R}{R_*})^a]^{\frac{2(6-\beta)}{9}}}, \tag{22}$$

where C is the integration constant.

If $H = H_*$ for $R = R_*$, we have a relation between constants A and C, is given by

$$C = H_*(1 + A)^{\frac{2(6-\beta)}{9}}. \tag{23}$$

Integrating (22) for $H = \frac{\dot{R}}{R}$, an expression for t in terms of the scale factor R, which for $a \neq 0$ is given by

$$Ct = \int \frac{[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2(6-\beta)}{9}}}{R} dR. \tag{24}$$

By defining $q = -(\frac{R\ddot{R}}{\dot{R}^2})$, it follows from (21) that during the course of evolution the deceleration parameter is given by

$$q = \frac{(6-\beta)\gamma - 3}{3}, \tag{25}$$

which is clearly depends upon R via γ. We solve (24) for two different early phases: *inflationary* and *radiation-dominated.*

3.0.1 Inflationary Phase

For inflationary phase ($R \ll R_*$), the second term on right hand side of integral in (24) dominates which gives the solution for scale factor R ($a \neq 0$) as,

$$R = R_* \left[\frac{2a(6-\beta)}{9} Ct\right]^{\frac{9}{2a(6-\beta)}}, \tag{26}$$

where (26) shows that during inflation, the dimensions of the universe increase according to law $R \propto t^{\frac{9}{2a(6-\beta)}}$, which is the case of power-law inflation and for the expanding universe, we must have $0 < \beta < 6$. Equation (26) also indicates that $R = 0$ at $t = 0$, $R \to \infty$ and $\dot{R} \to 0$ as $t \to \infty$. From (26), we have

$$H = \frac{9}{2a(6-\beta)} t^{-1}. \tag{27}$$

Substituting the value of H from (27) into (20), we get

$$\Lambda = \frac{81\beta}{[2a(6-\beta)]^2} t^{-2}. \tag{28}$$

Using (26) and (28) for $\gamma = \frac{2a}{3}$, (9) and (10) respectively, yield

$$\rho = D_1 t^{-\frac{12}{(6-\beta)}}, \tag{29}$$

$$G = \frac{81}{32\pi a^2(6-\beta)D_1} t^{\frac{2\beta}{(6-\beta)}} + G_0, \tag{30}$$

where $D_1 = R_*^{-\frac{8a}{3}} [\frac{2a(6-\beta)}{9} C]^{-\frac{12}{(6-\beta)}}$ and G_0 are the constants of integration for the consistency of the system. For physical significance $G > 0$ and $\rho > 0$, we must have $0 < \beta < 6$ and $D_1 > 0$. As t tends to zero, energy density becomes infinity and spatial volume is zero. For $\beta = 0$, it is observed that G = constant, $\rho \propto t^{-2}$ and $R \propto t^{\frac{3}{4a}}$. Putting the limiting value

$\gamma = \frac{2a}{3}$ in (25), the asymptotic value of deceleration parameter in the limit $\frac{R}{R_*} \ll 1$, is given by

$$q = \frac{2a(6-\beta)}{9} - 1. \tag{31}$$

This shows that the deceleration parameter is constant. Using (31), (27) can be written as

$$H = \frac{1}{1+q} t^{-1}. \tag{32}$$

From (30), we find

$$\frac{\dot{G}}{G} = \frac{2\beta}{(6-\beta)} t^{-1}. \tag{33}$$

Here, we observe that $\dot{G} \to 0$ as $t \to \infty$. In particular for $\beta = 2$ we get $G \propto t$, $\rho \propto t^{-3}$ and for $\beta = 3$ we have $G \propto t^2$, $\rho \propto t^{-4}$ which gives the result obtained earlier by Singh [27] in the context of general theory of relativity. From this results, we observe that the gravitational constant is always increases with time where as energy density decreases with time for $0 < \beta < 6$.

3.0.2 Radiation-dominated Phase

For radiation-dominated phase ($R \gg R_*$), the first term on right-hand side of the integral in (24) dominates which gives the solution for scale factor

$$R = R_* \left[\frac{4(6-\beta)}{9A^{\frac{2(6-\beta)}{9}}} C t \right]^{\frac{9}{4(6-\beta)}}, \tag{34}$$

From (34) we observe that $R = \propto t^{\frac{9}{4(6-\beta)}}$, which is the case of power-law expansion and for the expanding universe, we must have $0 < \beta < 6$. From (34), we find the following solutions for Hubble parameter, cosmological constant, energy density and gravitational constant, respectively,

$$H = \frac{9}{4(6-\beta)} t^{-1}, \tag{35}$$

$$\Lambda = \frac{81\beta}{16(6-\beta)^2} t^{-2}, \tag{36}$$

$$\rho = D_2 t^{-\frac{6a}{(6-\beta)}} \tag{37}$$

$$G = \frac{81}{128\pi(6-\beta)D_2} t^{\frac{6a-2(6-\beta)}{6-\beta}} + G_{00}, \tag{38}$$

where $D_2 = R_*^{-\frac{8a}{3}} [\frac{4(6-\beta)}{9A^{\frac{2(6-\beta)}{9}}} C]^{-\frac{6a}{(6-\beta)}}$ and G_{00} are the constants of integration for the consistency of the system. For energy density and gravitational constant to be positive, we must have the constant $D_2 > 0$ and $0 < \beta < 6$. This implies that G is always increasing with time. The spatial volume is zero at $t = 0$ and energy density tends to infinity as t tends to zero. We also observe that $\Lambda \propto t^{-2}$ which matches with its natural dimensions. Putting the limiting value $\gamma = \frac{4}{3}$ in (25) the asymptotic value of deceleration parameter in the limit $\frac{R}{R_*} \gg 1$, is given by

$$q = \frac{4(6-\beta)}{9} - 1, \tag{39}$$

which shows that the deceleration parameter is constant. Recently it has been found that the universe is probably accelerating at the present epoch. There are several justifications for this acceleration. Some authors attribute this acceleration to the presence of some scalar field has a considerable contribution to the total energy density of the present universe. The density parameter Ω_{rad} is given by

$$\Omega_{rad} = \frac{8\pi G\rho}{6H^2} = 1 - \frac{\beta}{6}. \tag{40}$$

Employing (35) and (36), this gives

$$\Omega_\Lambda = \frac{\beta}{6}. \tag{41}$$

If the universe is spatially flat, then we define Ω_{total} as

$$\Omega_{total} = \Omega_{rad} + \Omega_\Lambda. \tag{42}$$

Using (40) and (41) into (42), we get $\Omega_{total} = 1$. For $\beta = 0$, it is observed that $\rho \propto t^{-a}$, $R \propto t^{\frac{3}{8}}$ and $q = \frac{5}{3}$. Thus we find that the deceleration parameter varies from $q = \frac{2a(6-\beta)}{9} - 1$ at $R \ll R_*$ to $q = \frac{4(6-\beta)}{9} - 1$ for $R \gg R_*$. For $\beta = 6(1-a)$ $G \propto t^{1-}$, $\rho \propto t^{-1}$ and $R \propto t^{3/8a}$. Therefore, the declaration parameter is negative for positive cosmological constant ($\beta < \frac{15}{4}$) and for ($\beta > \frac{15}{4}$), the cosmological constant is negative.

4 Particular Solutions

We now study the solution in the limit $a \to 0$. For $a = 0$ the parameter γ varies from 0 for $R = 0$ to $\frac{4}{3}$ when $R \gg R_*$. In this case, (21) gives

$$H = \frac{H_i}{[A(\frac{R}{R_*})^2 + 1]^{\frac{2(6-\beta)}{9}}}, \tag{43}$$

where H_i is the initial value of H at $R = 0$. Integrating (43) for $H = \frac{\dot{R}}{R}$, we obtain

$$H_i t = \int \frac{[A(\frac{R}{R_*})^2 + 1]^{\frac{2(6-\beta)}{9}}}{R} dR. \tag{44}$$

Again, in the limit of very small R ($R \ll R_*$), the second term in the integral of (44) dominates and one has an exponential inflation, given by

$$R = R_* \exp(H_i t). \tag{45}$$

It is observed that as $t \to -\infty$, $R \to 0$ which shows the universe is infinitely old for $a = 0$ and have exponential inflation phase. The solution (45) is the familiar de-Sitter inflationary solution and in this case there exists an event horizon, which is given by

$$R_E = R_{t_0} \int_{t_0}^{\infty} \frac{dt}{R(t)} = \frac{1}{H_i}. \tag{46}$$

This limit is called the event horizon and has the value $\frac{1}{H_i}$. This implies that no observer beyond this proper distance at $t = t_0$ can communicate with another observer. The Hubble's constant gives the initial value H_i. It is also found that $\Lambda = 3\beta H_i^2$, $G =$ constant and $q = -1$.

On the other hand, in the limit of $R \gg R_*$ when the universe enters a radiation phase, the first term in integral of (44) dominates and the radiation phase is described by the solution

$$R = R_* \left[\frac{4(6-\beta)}{9} \frac{H_i}{A^{\frac{2(6-\beta)}{9}}} t \right]^{\frac{9}{4(6-\beta)}}. \tag{47}$$

The deceleration parameter varies from $q = -1$ at $R = 0$ to $q = \frac{4(6-\beta)}{9} - 1$ for $R \gg R_*$.

5 Conclusion

This work, has thus generalized to higher dimensions the well-known results in four dimensional space time. It is found that the difference is significant at least in the principal to the analogous situation in four dimensional space time. We have consider the higher dimensional Kaluza-Klein type cosmological model with varying gravitational and cosmological constants and discussed the problem by applying gamma law equation of state. We have obtained the solution in the framework of higher dimensional space time with the assumption $\Lambda \propto H^2$. The behavior of the scale factor, energy density ρ, gravitational constant G and cosmological constant Λ have been studied for two different phases: *inflationary and radiation*. It is observed that in both the cases the cosmological constant Λ retains the natural dimension with time i.e. $\Lambda \propto t^{-2}$ and gravitational constant G increases where as energy density decreases with time. The model is expanding in each case for $0 < \beta < 6$. The scale factor R evolves as $R \propto t^{\frac{9}{2a(6-\beta)}}$ in inflationary phase and $R \propto t^{\frac{9}{4(6-\beta)}}$ in radiation phase.

It is also shown that if universe is spatially flat, the total density parameter $\Omega_{total} = \Omega_{rad} + \Omega_\Lambda$ gives unity. This implies that the cosmological constant supplies the "missing matter" requires to make $\Omega_{total} = 1$ as suggested by the inflationary models, though on the basis of little observational evidence. A particular case (in the limit $a \to 0$) has also been examined in which scale factor varies exponentially with time as approaching zero for inflationary phase. The parameter γ varies 0 to $\frac{4}{3}$ when $R \gg R_*$. The possibility that $q = -1$ i.e. $q < 1$ has come in this case, which indicates that the universe is accelerating. In this way, unified description of early evolution of the universe is possible with variable gravitational and cosmological "constants" in the context of higher dimensional space time.

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